Expressions and Equations **8.EE**

Analyze and solve linear equations and pairs of simultaneous linear equations.

7. Solve linear equations in one variable.
   a. Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).

The basis for understanding this standard is when students first solve simple equations. Students do this informally as early as Kindergarten when finding a number that can be added to a given number to make 10. More formal equation-solving starts in grade 6 when students reason about sets of values that make an equation true, and create and solve equations using abstract symbols.

The key to this standard is knowing a few properties of equality. Specifically:

- reflexive property of equality: $a = a$
- addition property of equality: $a = b$ if and only if $a + c = b + c$
- multiplication property of equality: $a = b$ if and only if $ac = bc$, when $c \neq 0$

Let's begin with an equation, like $4x-6=2x+8$. If we try substituting some values for x, we find that 7 makes the true statement 22 = 22 (reflexive property).

$$\begin{aligned} 4(7)-6 &= 2(7)+8 \\ 28-6 &= 14+8 \\ 22 &= 22 \checkmark \end{aligned}$$

Mathematical Practices
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Let's *transform* the equation to into a simpler form using the addition property of equality.

$$\begin{aligned} 4x-6 &= 2x+8 \\ 4x-6+(-2x) &= 2x+8+(-2x) \\ 2x-6 &= 8 \end{aligned}$$

If we substitute values again, we still find that 7 gives us the true statement 8 = 8.

$$\begin{aligned} 2(7)-6 &= 8 \\ 14-6 &= 8 \\ 8 &= 8 \checkmark \end{aligned}$$

Transforming the equation did not change its solution. Further transforming it with the addition property of equality, we get

$$\begin{aligned} 2x-6 &= 8 \\ 2x-6+6 &= 8+6 \\ 2x &= 14 \end{aligned}$$

which is true when $x = 7$, and then using the multiplication property of equality,

$$\begin{aligned} 2x &= 14 \\ \left(\frac{1}{2}\right)2x &= \left(\frac{1}{2}\right)14 \\ x &= 7 \end{aligned}$$

If we substitute 7 in for x, we get 7 = 7, which is a true statement. There's that reflexive property again.

We *transformed* the equation into progressively simpler forms and each time we could make a true statement by substituting 7 in for x. Students should conceptually know by grade 8 that such transformations do not change the solution to the equation. What may not be clear to them is that we transform equations to get to the form $x = a$, because that gives us the solution to the original equation.

What if we begin with the equation $4x-6=2(2x+8)$? We could substitute all day long and not find anything that works. (We need students to try—maybe not all day, but at least for a few minutes.)

If we transform the equation to into simpler and simpler forms, we get

$$\begin{aligned} 4x-6&=2(2x+8) \\ 4x-6&=4x+16 \\ -6&=16 \; \times \end{aligned}$$

In the second line, we still can't substitute anything for *x* that makes the statement true. The third line tells the whole story—it is flat out false: $-6 \neq 16$! If it were true, it would be saying $a = b$, where *a* and *b* are different numbers. That is impossible. Remember, if there were a solution to the original equation, transforming the equation wouldn't change that. But the last line is never true; therefore, the original equation is never true. The original equation has no solution.

Finally, consider the case of $4x-6=2(2x-3)$. What can we substitute for *x* that makes the statement true? Well, $x = 1$ works, as does $x = 2$, $x = 10$, and $x = –3.14$. In fact, every one of you reading this could come up with a different answer and there would still be an infinite number of choices left over. Transforming the equation,

$$\begin{aligned} 4x-6&=2(2x-3) \\ 4x-6&=4x-6 \\ -6&=-6 \; \checkmark \end{aligned}$$

If we substitute values into the second line, anything we try works. The last transformation reveals the situation: –6 = –6 is a true statement. But, we made it to that true statement, $a = a$, without substituting anything. Since the final transformed equation is *always* true, the original equation is always true regardless of the value of *x*. The original equation has an infinite number of solutions.

Eventually, students will be able to recognize these situations earlier than the final form. The equation $4x – 6 = 4x + 16$ should be clue enough that there is no solution, since subtracting 6 from, and adding 16 to, the same number (4*x*) cannot possibly result in a true statement. Likewise, $4x – 6 = 4x – 6$ shows reflection, so transforming further isn't really necessary.

When assessing this standard, teachers will want students to do a couple things. First, they should be able to describe the number of solutions a linear equation has by transforming it to one of the three cases: $x = a$ (one solution), $a = a$ (infinite solutions), and $a = b$ (no solution). Additionally, they should be able to justify why those transformations do not change the solution, if one exists, or the number of solutions to the original equation. Finally, students should be able to construct their own equations, first by taking one of the final forms and transforming it into a more complex linear equation, then creating one without transformations by using their understanding of the structure of linear equations and the number of solutions they can have.

Understanding this concept helps with later topics like systems of linear equations in two variables. When solving a system algebraically, one gets similar outcomes to those discussed. Consider the two situations below. Each is solved by substituting the right hand side of the first equation into the second equation for *y*.

$$\begin{cases} y=4x-6 \\ 2y=8x-12 \end{cases} \qquad \begin{cases} y=4x-6 \\ 2y=8x-6 \end{cases}$$

$$\begin{aligned} 2(4x-6)&=8x-12 \\ 8x-12&=8x-12 \\ -12&=-12 \; \checkmark \end{aligned} \qquad \begin{aligned} 2(4x-6)&=8x-6 \\ 8x-12&=8x-6 \\ -12&=-6 \; \times \end{aligned}$$

In the first case, we get the form $a = a$, so there are an infinite number of solutions. The two equations are different representations of the same line. In the second case, we get the form $a = b$, so there is no solution; the two lines are parallel.

Connections

6.EE.2–4
6.EE.7
7.EE.1–2
7.EE.4
8.EE.8
8.F.1–3
A.SSE.1–3
A.APR.4
A.REI.1–11
F.IF.1
F.IF.3

We have seen that with linear equations in one variable we can transform them until one of three simple cases occurs. Is it that way with other types of equations? The answer is no, but it depends on the circumstances. When students take higher-level classes, they will find that doing certain transformations may result in losing a solution or gaining an extraneous solution. There are methods to check for those situations. For now, students should be safe in knowing their transformations of linear equations will correctly lead them to one solution, no solution, or an infinite number of solutions.